9. L. I. Sedov, Continuum Mechanics [in Russian], Vol. 2, Moscow (1984).
10. K. Sensenig, "Certain problems on buckling in the nonlinear theory of elasticity," in: Theory of Buckling and Nonlinear Eigenvalue Problems [Russian translation], Mir, Moscow (1974).
11. L. M. Zubov, "Approximate theories of the buckling of thin plates of a semi-linear material with an affine initial configuration," Prikl. Mat. Mekh., 33, No. 4 (1969).
12. L. M. Zubov, "Buckling of plates of a non-Hookian material with affine initial deformation," ibid., 34, No. 4 (1970).
13. A. A. Zelenin and L. M. Zubov, "Behavior of a thick circular plate after loss of stability," Prikl. Mat. Mekh., 52, No. 4 (1988).
14. K. Hatter and F. Williams, "Theory of ice floes," in: Physics and Mechanics of Ice [Russian translation], Mir, Moscow (1983).

BEGINNING OF PLASTIC YIELDING IN A STRESS CONCENTRATION ZONE

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Classical strength criteria are currently being widely used in the strength design of structural elements. Here, it is assumed that plastic yielding begins when, in accordance with the chosen criterion, the limiting stress state is attained at even one (the most heavily stressed) point of the structure. However, these criteria do not always consider how the beginning of plastic flow is affected by the nonuniformity of the stress distribution near the point of greatest stress.

The subject of the effect of nonuniformity of the stress state on the yield point in the region where the stresses are maximal has long been of interest to researchers [1-3]. Subsequent to [1-3], investigators made use of the gradient approach proposed in [1] to evaluate this nonuniformity and its effect on the local yield point at the most heavily stressed point of the body [4-6]. Signs of plastic flow in the region of maximum stresses were considered to be the appearance of Lüders' lines in specimens of mild steel [1] and deviations from elastic strain laws detected by strain gauges or other means [4, 5]. It was noted that these indications of plastic yielding are manifest when the stresses at the most heavily stressed point exceed the yield point in a uniform stress state σ_y. Recent experiments have detected deviations from elastic strain laws by the highly sensitive method of holographic interferometry [7, 8]. These experiments have also confirmed that there is an increase in the local yield point at the most heavily stressed point of the body. The results that were obtained were used as a basis for proposing a gradient criterion for the onset of plastic flow in a nonuniform stress state [9-11].

In the present study, we use the example of the tension of a plate with an elliptical hole to examine the range of validity of the gradient criterion and the continuum model in the case of very small holes. We note that there is a connection between this criterion and the structure of the material, and we show that the criterion actually reflects the energy dependence of the beginning of plastic flow for a fairly broad range of stress-concentration factors and hole sizes.

1. Range of Validity of the Gradient Criterion and the Continuum Model in the Case of Very Small Holes. In accordance with the gradient criterion, in a nonuniform stress state plastic strains occur only when an equivalent stress – let this be the stress intensity σ_i – at the most heavily stressed point of the given body σ_i^{max} exceeds σ_y and reaches the local yield point σ_y^{ℓ}:

$$\sigma_y^{\ell} = \sigma_y\left(1 + \sqrt{L_0 G/\sigma_i^{\max}}\right). \tag{1.1}$$

Here, $G = |\mathrm{grad}\,\sigma_i|$ is the modulus of the gradient of σ_i at the point subject to the greatest

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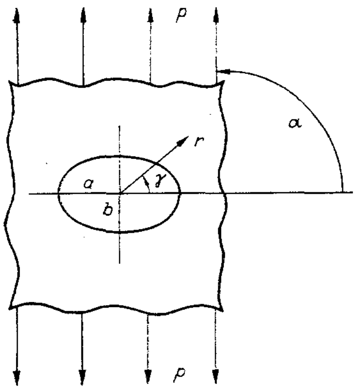

Fig. 1

stress; L_0 is a material constant having the dimension of length. In a nonuniform stress state, $G = 0$ and $\sigma_y^\ell = \sigma_y$. This criterion is substantiated by experiments involving the tension of wide flat specimens with a central elliptical hole and specimens with lateral notches [9-11]. In [9-11], the stress-intensity factor ranged in value from 3 to 10, while the dimension of the elliptical hole in the direction perpendicular to the tension direction was 10 or 20 mm. However, it must be noted that, for smaller holes, the radius of curvature across the specimen at the tip of the concentrator will also be smaller. According to [10], at finite values of K, the radius of curvature at the tip of the concentrator also cannot be less than a certain value if contradictions are to be avoided in the model.

This assertion requires additional study. Our goal here is to determine the range of application of the gradient yield criterion for small holes. The fact is that G and σ_y^ℓ increase without limit when the dimensions of the hole in the section liable to the greatest stress decrease to infinitely small values. Thus, in the case of very small holes, σ_i^{max}, while remaining at or below the local yield point, may take such exaggerated values that the nominal stress away from the hole, connected with σ_i^{max} by the elastic solution, will exceed σ_y, i.e., the gradient criterion of the beginning of plastic flow will become physically meaningless.

Let us examine this contradiction using the example of the tension of a plate with an elliptical hole (Fig. 1) when $\alpha = 0$ and $\pi/2$. In Fig. 1a and b are the major and minor semi-axes of the ellipse and α is the angle between the major semi-axis and the tension direction. Knowing the elastic solution [12], we can obtain an expression for the modulus of the gradient of σ_i at the tip of the concentrator:

$$G = \sigma_i^{max}(1{,}25 + 0{,}5/K)(K - 1)^2/C. \tag{1.2}$$

Here, K is the stress concentration factor ($K = 1 + 2a/b$ at $\alpha = \pi/2$, $K = 1 + 2b/a$ at $\alpha = 0$); C is the dimension of the hole in the section liable to the greatest stress ($C = 2a$ at $\alpha = \pi/2$, $C = 2b$ at $\alpha = 0$). For sufficiently small C and $K \neq 1$, in accordance with (1.1) and (1.2) we will have an exaggerated value of the local yield point σ_y^ℓ such that it will reach the value σ_y at infinity before it reaches the value of σ_y^ℓ at the tip of the concentrator. Since plastic flow has not yet begun by this moment, we can use the elastic solution and write a condition expressing the impermissibility of such a situation:

$$\sigma_y^\ell \leqslant K\sigma_y. \tag{1.3}$$

In essence, this condition means that plastic flow at the tip of the concentrator cannot begin later than it does away from the hole. It follows from this condition and Eq. (1.1) that $\sqrt{L_0 G/\sigma_i^{max}} \le K - 1$. Squaring both sides of this inequality and inserting Eq. (1.2) for G, we obtain $L_0(1.25 + 0.5/K)(K - 1)^2/C \le (K - 1)^2$. From this, we find the desired limitation $C \ge (1.25 + 0.5/K)L_0$ or $K = 1$.

Thus, in the tension of a plate with an elliptical hole, the gradient criterion can be used without violating condition (1.3) only if the dimension of the hole in the section liable to the greatest stress is no less than

$$C_* = (1{,}25 + 0{,}5/K)L_0. \tag{1.4}$$

With a change in K from 1 (for cracks extending in the tension direction) to an infinitely large value (for cracks perpendicular to the tension direction), C_* changes only slightly (from $1.75L_0$ to $1.25L_0$). For a circular hole, K = 3 and $C_* = (4.25/3)L_0$. It should be noted that $L_0 = 0.16$ mm was found in [9-11] in experiments conducted for materials D19AT, V95, and St. 3. Thus, for these materials, condition (1.3) is violated only for very small holes in the most heavily stressed section. Here, the condition is violated when the dimensions of the hole lie within the range 0.20-0.28 mm – the exact value required depending on the shape of the hole.

We will assume that the contradiction discussed above is a consequence of having used the continuum model to describe the properties of the material while attempting to explore the question of the onset of plastic flow in a body having a very small hole. If we suppose that a material already contains microcracks of length $1.25L_0$ or other defects equivalent with respect to the local yield point when we create small holes with a dimension less than C_* in the section liable to the greatest stress, then according to the gradient criterion plastic flow will begin near the tips of the numerous natural defects when the nominal stress reaches the value σ_y. Such a flow will in turn lead to the plastic deformation normally expected to occur in the material in a uniform stress state. Thus, the contradiction is resolved. A similar hypothesis explaining the actual strength of brittle materials in terms of microcracks already present in them was advanced in the classical work on fracture mechanics [13].

If we speak of local ultimate strength rather than local yield point for brittle materials, it becomes interesting, in the light of this hypothesis, to examine the experimental data on the static strength of notched cast-iron specimens whose internal structure is characterized by different degrees of nonuniformity [14]. Serensen and Kramarenko [14] concluded on the basis of this data that the effect of a stress concentration on static strength is greatest for the strongest and most uniform cast irons. Thus, other conditions being equal, the smaller the internal discontinuities, the smaller the increase in local yield strength.

Serensen and Kramarenko [14] studied brittle fracture; the increase in local ultimate strength was represented as a function of the modulus of the gradient of the first principle stress σ_1 divided by σ_1^{max}. Thus, the smaller the internal discontinuities, the smaller the value of the parameter L_1 (analogous to L_0) with the modulus of the gradient of σ_1. It should be noted that the mean fracture stress for notched cast-iron specimens depends only slightly on the parameter K and is close to the fracture stress for smooth specimens. Thus, in order to describe experimental data on the strength of cast-iron structural elements using the gradient criterion, it is necessary to take values of L_1 that are considerably greater than the values of L_0 found for materials D19AT, V95, and St. 3. This is a consequence of the high degree of nonuniformity of the internal structure of cast irons compared to these materials.

In contrast to cast iron, the strength of a homogeneous material such as glass depends very heavily on the presence of stress raisers (even small scratches), i.e., from the viewpoint of the gradient approach, L_1 is very small. However, as was shown by Griffith [13], very small (depth of about 10^{-3} mm) scratches on the surface of glass do not have a weakening effect – although the stress concentration at the bottom of the scratch is very high. Griffith attributed this to the presence of numerous microscopic cracks, comparable in size to the depth of the scratch, in glass in its natural state.

Thus, we see that there is a certain connection between the gradient criterion and the structure of a material. The existence of this connection makes it possible to establish specific sizes of stress concentrators for which the continuum model and the nonclassical gradient criterion of the limiting state of a material in a nonuniform stress state can be used to determine the beginning of plastic flow or fracture of actual structures. Within the framework of the continuum model, the above-noted contradiction renders the gradient criterion invalid for concentrators smaller than a certain size determined on the basis of the criterion from experiments conducted on specimens with large concentrators.

2. Energy Analysis of the Beginning of Plastic Flow. In accordance with gradient criterion (1.1), by the moment of the beginning of plastic flow at those points near the stress raiser where $\sigma_i > \sigma_y$, the specific elastic strain energy will be greater than its critical value (which corresponds to the condition $\sigma_i = \sigma_y$). The specific strain energy is expressed

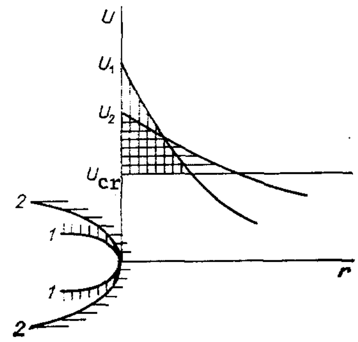

Fig. 2

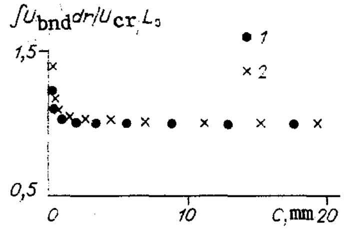

Fig. 3

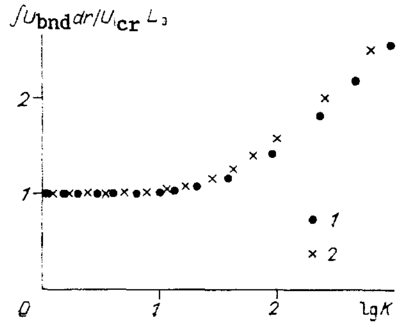

Fig. 4

by the formula [15] $U = ((1+\mu)/3E)\sigma_i^2$ (μ is the Poisson's ratio and E is the Young's modulus). At $\sigma_i = \sigma_y$, it has the critical value $U_{cr} = ((1+\mu)/3E)\sigma_y^2$. Then the specific excess energy at $\sigma_i > \sigma_y$ will be

$$U_{bnd} = U - U_{cr} = \frac{1+\mu}{3E}(\sigma_i^2 - \sigma_y^2).$$

In the problem of the tension of a plate with an elliptical hole (see Fig. 1) with α = 0 and $\pi/2$, we obtained some interesting results by calculating the integral of specific excess energy in that part of most heavily stressed section where $\sigma_i > \sigma_y$ by the beginning of plastic flow. It turned out that, within a fairly broad range of hole sizes and shapes, the integral nearly has a constant value. This value is in turn the product of two material constants, i.e., is itself a material constant:

$$\int\limits_{\sigma_i>\sigma_y} U_{bnd}dr = \frac{1+\mu}{3E}\int\limits_{\sigma_i>\sigma_y}(\sigma_i^2 - \sigma_y^2)\,dr \approx U_{cr}L_0. \tag{2.1}$$

The results obtained here have confirmed the original proposition that since plastic flow in actual solids begins within a certain volume of a material rather than at a certain point, then the mathematical condition for its inception must also be sought in a certain neighborhood of the point liable to the greatest stress in the body. On diagrams showing the distribution of specific excess energy across the most heavily stressed section for two different concentrators (Fig. 2), Eq. (2.1) can be represented as follows: the vertically hatched area [the value of the integral in (2.1) for concentrator 1] is nearly equal to the horizontally hatched area (the value of the integral for concentrator 2).

The accuracy and range of validity of Eq. (2.1) can be judged from Figs. 3 and 4, which show the results of calculation of the normalized integral of specific excess energy

$$\int\limits_{\sigma_i>\sigma_y} U_{bnd}dr/U_{cr}L_0.$$

The value $L_0 = 0.16$ mm, obtained from experiments, was used in these calculations. It is evident from Fig. 3, showing values of the normalized integral for circular ($K = 3$) and elliptical ($K = 6$) holes (points 1 and 2) of different sizes, that Eq. (2.1) is satisfied quite well on the whole. However, the values of the integral exceed the nominal value for cases in which the dimensions of the hole in the most heavily stressed section are close to C_*, i.e., are close to the limits of the validity of condition (1.3). This is to be expected, since at $C < C^*$ the gradient criterion gives clearly exaggerated values of the nominal stresses corresponding to the beginning of plastic flow. It is evident from Fig. 4, showing the dependence of the normalized integral on the stress concentration factor K for two fixed hole dimensions across the most heavily stressed section $C = 20$ and 10 mm (points 1 and 2), that for moderate concentration factors ($K < 10$) the integral takes nearly constant values. At very large K, the values of the integral increase significantly.

The results shown here indicate that the gradient criterion actually reflects the energy dependence of the beginning of plastic flow within a fairly broad range of stress concentration factors and hole dimensions. In accordance with this criterion, plastic strains occur only when the integral of specific excess energy reaches the limiting value $U_{cr}L_0$ in that part of most heavily stressed section in which $\sigma_i > \sigma_y$.

At the same time, the integral of specific excess energy calculated in accordance with the gradient criterion exceeds $U_{cr}L_0$ for concentrators with a crack-like shape and for small holes. Thus, in these cases, the nominal stresses at which plastic flow begins are lower than the values predicted by the gradient criterion. Further study is needed to establish the degree of agreement between the energy model of the beginning of plastic flow and the gradient criterion for limiting cases. For example, the requirement that the values of the integral be constant in the case of small holes not only leads to a decrease in the limiting nominal stresses corresponding to the beginning of plastic flow, it also leads to a decrease — relative to Eq. (1.4) — in the dimensions C_* at which these nominal stresses exceed σ_y. Thus, new (lower) estimates of the dimensions of natural stress concentrators in a material can be obtained. The values of the integral exceed the nominal value for very large values of K because the actual stress distribution in a material near the tip of a crack-like concentrator far from always corresponds to the classical solution of the linear problem of elasticity.

Thus, according to this solution, enormous values of stress intensity and specific strain energy are obtained for sufficiently small but finite nominal loads near the tip of a crack-like concentrator. However, allowing for geometric nonlinearity and the deviations from Hooke's law in the problem of the tension of a plate with a crack-like concentrator leads to a reduction in the values of these quantities in the neighborhood of the tip of the concentrator for the same nominal load [16]. Thus, the values of the integral of specific excess energy, significantly exceeding $U_{cr}L_0$, that are obtained for very large K do not correspond to the actual value of the integral for real materials.

Consequently, our analysis has shown that the criterion of the beginning of plastic flow in a nonuniform stress state is of an energy nature and can be expressed fairly simply in energy units, assuming the latter are correctly calculated. In particular, in the case of the tension of a plate with an elliptical hole, plastic strains occur only when the integral of specific excess energy reaches the limiting value $U_{cr}L_0$ in that part of the most heavily stressed section where $U > U_{cr}$.

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LITERATURE CITED

1. N. N. Davidenkov and G. P. Zaitsev, "Mechanical analysis of impact brittleness," Zh. Tekh. Fiz., 2, No. 5 (1932).
2. W. Kuntze, "Ermittlung des Einflusses ungleichformiger Spannungen und Querschnitt auf die Streckgrenze," Stahlbau, 6, No. 7 (1933).
3. S. V. Serensen, Strength of Metal and Design of Machine Parts [in Russian], ONTI NKTP, Moscow, Leningrad (1937).
4. A. Bascoul and J. C. Maso, "Influence des gradients de contraintes ou de deformation sur le seuil d'écoulement plastique d'un acier," J. Mec. Appl., 5, No. 4 (1981).

5. V. A. Baldin, V. N. Potapov, and A. A. Fadeev, "Resistance of steel to deformation with a nonuniform stress distribution," Stroit. Mekh. Raschet Sooruzh., No. 5 (1982).
6. G. A. Geniev and S. Yu. Kalashnikov, "Effect of stress gradients and the geometric scales of the cross sections on the transition of bent elements to the plastic state," in: Studies in Structural Mechanics: Collection of Scientific Transactions [in Russian], TsNIISK, Moscow (1985).
7. A. M. Ivanov and M. D. Novopashin, "Method of studying the formation of plastic zones near concentrators during elastoplastic deformation," in: Strength of Structural Materials and Elements under Low-Temperature Conditions. Collection of Scientific Transactions [in Russian], Yakutsk (1985).
8. M. D. Novopashin and A. M. Ivanov, "Calculation of the load-carrying capacity of structural elements with a nonequilibrium stress distribution," in: Increasing the Cold Resistance and Load-Carrying Capacity of Structures. Collection of Scientific Transactions [in Russian], Yakutsk (1987).
9. M. D. Novopashin and S. V. Suknev, "Gradient criterion of the yielding of elements of metallic structures with stress concentrators," in: Modeling in Engineering. Collection of Scientific Transactions [in Russian], Vol. 1(18), No. 3, Novosibirsk (1987).
10. M. D. Novopashin and S. V. Suknev, "Gradient criterion of the local yielding of elements of metallic structures with stress concentrators," Preprint, Akad. Nauk SSSR Sib. Otd., IFTPS, Yakutsk (1987).
11. M. D. Novopashin, "Elastoplastic deformation and the limiting state of structural elements with stress concentrators," Author's Abstract of Engineering Sciences Doctoral Dissertation, Novosibirsk (1988).
12. N. I. Muskhelishvili, Some Fundamental Problems of the Mathematical Theory of Elasticity [in Russian], Nauka, Moscow (1966).
13. A. A. Griffith, "The phenomena of rupture and flow in solids," Philos. Trans. R. Soc. London, Ser. A, 221, 163 (1921).
14. S. V. Serensen and O. Yu. Kramarenko, "Structural strength of nodular cast iron," Vestn. Mashinostr., No. 1 (1959).
15. G. S. Pisarenko and A. A. Lebedev, Deformation and Strength of Materials under Complex Stress [in Russian], Naukova Dumka, Kiev (1976).
16. L. I. Sedov, Continuum Mechanics [in Russian], Vol. 2, Nauka, Moscow (1970).

INSTABILITY OF ELASTOPLASTIC PLANE FLOWS

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Under conditions of high-speed elastoplastic deformation material flow inhomogeneities associated both with the presence of elastic forces, which may cause self-oscillating processes, and with localized adiabatic heating on a narrow interval of the highest strain rates, are found to occur [1]. Thermoplastic shear was investigated mathematically in [2, 3], and a model of an elastoviscous fluid was considered in [4]. Here, the case of plane elastoplastic flow is studied with allowance for the thermal effects associated with adiabatic conditions and the convective removal of heat from the zone of intense deformation processes.

1. The equation of motion of the medium and the energy balance equation take the form [3, 4]:

$$\frac{\partial V}{\partial T} + V_C \frac{\partial V}{\partial Y} = \frac{1}{\rho} \frac{\partial S}{\partial Y}; \tag{1.1}$$

$$\rho c_V \left(\frac{\partial \Theta}{\partial T} + V_C \frac{\partial \Theta}{\partial Y} \right) = \lambda \frac{\partial^2 \Theta}{\partial Y^2} + \beta S \frac{\partial \Gamma}{\partial T}, \tag{1.2}$$

where V, S, Θ, and Γ are the velocity, stress, temperature, and degree of deformation, respectively, Y is a coordinate, T is time, V_C is the convective velocity component, ρ, c_V, and λ are the density, specific heat, and thermal conductivity coefficient, and β = 0.9-0.95 is a coefficient.

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